CONSTRUCTION OF CONTROL CHARTS

In this lecture we are going to discuss the construction of control charts for variables. The total lecture is subdivided in to

1. **Data and necessary statistical computation**
2. **General model of control limits**
3. Construction of **limits for** $\overline{X}$, R and S - Chart
4. **Steps in constructing** $\overline{X}$ **Chart, R-Chart (or S-Chart):**
5. Interpretation of $\overline{X}$ **Chart, R-Chart**
6. Table of constants
7. **Conclusion**

1. **Data and necessary statistical computation:**

Let us assume that k samples of size n each is taken with observations X_1, X_2, . . . , $X_{n.}$ The collected sample information will form the data set as under:

Table-1: Sample data

Sample Number	1	2	3	. . .	n	$\overline{X_i}$	R	s
1	X_{11}	X_{12}	X_{13}	. . .	X_{1n}	$\overline{X_1}$	R_1	s_1
2	X_{21}	X_{22}	X_{23}	. . .	X_{2n}	$\overline{X_2}$	R_2	s_2
3	X_{31}	X_{32}	X_{33}	. . .	X1n	$\overline{X_3}$	R_3	s_3
4	X_{41}	X_{42}	X_{43}	. . .	X_{4n}	$\overline{X_4}$	R_4	s_4
5	X_{51}	X_{52}	X_{53}	. . .	X_{5n}	$\overline{X_5}$	R_5	s_5
. . .	. . .	. . .	. . .	. . . .	. . .	. . .	. . .	. . .
k	X_{k1}	X_{k2}	X_{k3}	. . .	X_{kn}	$\overline{X_k}$	R_k	s_k

a) Calculate the overall process average

$\overline{\overline{X}} = \frac{\overline{X_1} + \overline{X_2} + ... + \overline{X_k}}{k}$, where $\overline{X_i}$ is the average of each subgroup and is given by

$$\overline{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

b) Estimation of σ using sample ranges - from the sampling distribution of sample range R, it can be derived that E(R)=$d_2\sigma$ and SE(R)=$d_3\sigma$ where d_2 and d_3 are constants depending upon the sample size n. The constants d_2 and d_3 are obtained based on the approximate relationship in finding the expected value of R in the distribution of R as the distribution of R depends on n, the sample size.

Hence an estimate of $\boldsymbol{\sigma}$ using sample ranges can be computed as $\hat{\sigma} = \frac{\overline{R}}{d_2}$, where $\overline{R}$ is calculated as $\overline{R} = \frac{R_1 + R_2 + ... + R_k}{k}$ where R= X_{max}-X_{mim}

c) Estimation of σ using sample standard deviations - from the sampling distribution of sample standard deviation 's' , it can be derived that $E(s) = c_2\sigma$ and $SE(s) = c_3\sigma$ where c_2 and c_3 are constants depending upon the sample size n. The constants c_2 and c_3 are obtained based on the approximate relationship in finding the expected value of 's' in the distribution of s as the distribution of s depends on n, the sample size.

Hence an estimate of **σ** using sample standard deviations can be computed as $\hat{\sigma} = \frac{\bar{s}}{c_2}$,

where $\bar{s}$ is calculated as $\bar{s} = \frac{s_1 + s_2 + \ldots\ldots + s_k}{k}$ where $s^2 = \frac{\sum_{i-1}^{n}(x_i - \bar{x})^2}{n-1}$

2. General model of control limits:

Let T be a sample statistic that measures some quality characteristic of interest. Suppose that the mean of T is μ_T and the standard deviation of T is σ_T. Then the general model of control limits are given by,

Upper Control Limit (UCL)= $\mu_T + 3\sigma_T$

Center Line (CL) = μ_T

Lower Control Limit (LCL) = $\mu_T - 3\sigma_T$

Types of Control Charts

The Control Charts are classified depending on the nature of quality parameters like - measurable (e.g., length, resistance, diameter etc.) and non-measurable (e.g., good or bad, percentage of nonconforming units, number of nonconformities, etc.) as Variable Control Charts and Attribute's Control Charts.

CONTROL CHARTS FOR VARIABLES

Measurements on any quality characteristic from a process working under a system of chance causes are expected to follow the Normal distribution. About 99.73% of observations from a Normal population (with average μ and standard deviationσ) fall within μ ± 3σ. This knowledge is used to establish control limits at ±3σ from the central value of the process (representing the Central Line). Successive observations from this process can be plotted on a chart containing a Central line at μ and control limits μ + 3σ and μ - 3σ. Any observation falling outside the limits is very unlikely when only chance causes are operating on the process (having a probability of 0.0027); hence an event of this kind indicates presence of assignable causes.

3. Construction of **limits for** $\overline{X}$, R and S - Chart

The control chart for the process average is $\overline{X}$ -Chart. With the date set defined as above, the control limits for $\overline{X}$ -Chart are established as from the general model of control limits. Let T denote the process average observed by sample means ($\overline{X_i}$) at regular intervals of time.

Upper Control Limit (UCL) = $\mu_T + 3\sigma_T = \mu + \frac{3\sigma}{\sqrt{n}}$, since $E(\overline{X}) = \mu$ and $\sigma_T = \frac{\sigma}{\sqrt{n}}$

Center Line = $E(\overline{X}) = \mu$

Lower Control Limit (LCL) = $\mu_T - 3\sigma_T = \mu - \frac{3\sigma}{\sqrt{n}}$, since $E(\overline{X}) = \mu$ and $\sigma_T = \frac{\sigma}{\sqrt{n}}$

The control limits for $\overline{X}$ chart depend on the parameters μ and σ. Depending on whether μ and σ are known or estimated from the samples taken from the process, the control limits will take different structures as follows.

Case 1: When μ and σ are known and are assumed to take the values μ and σ.

Upper Control Limit (UCL) $= \mu + \frac{3\sigma}{\sqrt{n}}$

Center Line $= \mu$

Lower Control Limit (LCL) $= \mu - \frac{3\sigma}{\sqrt{n}}$

OR

Upper Control Limit (UCL) $= \mu + A\sigma$

Center Line $= \mu$

Lower Control Limit (LCL) $= \mu - A\sigma$ where the constant A is $\frac{3}{\sqrt{n}}$

Case 2*:* When μ and σ are unknown. In this case μ is estimated by $\overline{\overline{X}}$ and σ is estimated using the sample ranges as $\hat{\sigma} = \frac{\overline{R}}{d_2}$. Then for the $\overline{X}$-Chart, the control limits are:

Upper control limit $UCL_{\overline{X}} = \overline{\overline{X}} + \frac{3}{\sqrt{n}}\frac{\overline{R}}{d_2} = \overline{\overline{X}} + A_2\overline{R}$ and,

Central line = $CL_{\overline{X}} = \overline{\overline{X}}$

Lower control limit $LCL_{\overline{X}} = \overline{\overline{X}} - \frac{3}{\sqrt{n}}\frac{\overline{R}}{d_2} = \overline{\overline{X}} - A_2\overline{R}$

where $A_2 = \frac{3}{d_2\sqrt{n}}$ is a constant tabulated for different values of ‘n’.

Case 3: When μ and σ are unknown. In this case μ is estimated by $\overline{\overline{X}}$ and σ is estimated using the sample standard deviation as $\hat{\sigma} = \frac{\overline{s}}{c_2}$. Then for the $\overline{X}$-Chart, the control limits are:

Upper control limit $UCL_{\overline{X}} = \overline{\overline{X}} + \frac{3}{\sqrt{n}}\frac{\overline{s}}{c_2} = \overline{\overline{X}} + A_1\overline{s}$ and,

Central line = $CL_{\overline{X}} = \overline{\overline{X}}$

Lower control limit $LCL_{\overline{X}} = \overline{\overline{X}} - \frac{3}{\sqrt{n}}\frac{\overline{s}}{c_2} = \overline{\overline{X}} - A_1\overline{s}$

where $A_1 = \frac{3}{c_2\sqrt{n}}$ is a constant tabulated for different values of ‘n’.

R- Chart

The R-Chart is used for control of process variability when the process dispersion is measured using ranges (refer table-1). If sample Range is used to measure the process variability the control limits are:

Upper Control Limit (UCL) = E(R) +3SE(R)
Center Line = E(R)
Lower Control Limit (LCL) = E(R) -3SE(R)

From the sampling distribution of R it is known that $E(R) = d_2\sigma$ and $SE(R) = d_3\sigma$. Accordingly the control limits are simplified as:

Case 1: **σ known**

Upper Control Limit UCL_R $= (d_2 + 3d_3)\,\sigma = D_2\sigma$ where $D_2 = d_2 + 3d_3$
Central Line CL_R $= d_2\sigma$
Lower Control Limit LCL_R $= (d2 - 3d_3)\,\sigma = D_1\,\sigma$ where $D_1 = d_2 - 3d_3$

Case 2: **σ unknown**

When σ is unknown, it is replaced by its estimate $\frac{\overline{R}}{d_2}$. Then the control limits are

Upper Control Limit UCL_R $= (d_2 + 3d_3)\,\frac{\overline{R}}{d_2} = D_4\overline{R}$

Central Line CL_R $= d2\sigma = d_2\,\frac{\overline{R}}{d_2} = \overline{R}$

Lower Control Limit LCL_R $= (d_2 - 3d_3)\,\frac{\overline{R}}{d_2} = D_3\overline{R}$

where, $D_4 = (d_2 + 3d_3)/d_2$ and $D_3 = (d_2 - 3d_3)/d_2$

4. Construction of **S chart**

S -Chart:

The S-Chart is used for control of process variability when the process dispersion is measured using standard deviations (refer table-1). If sample standard deviation is used to measure the process variability the control limits are:

Upper Control Limit (UCL) = E(s) +3SE(s)
Center Line = E(s)
Lower Control Limit (LCL) = E(s) -3SE(s)

From the sampling distribution of 's' it is known that $E(s) = c_2\sigma$ and $SE(s) = c_3\sigma$. Accordingly the control limits are simplified as:

Case 1: **σ known**

Upper Control Limit UCL_s $= (c_2 + 3c_3)\sigma = B_2\sigma$ where $B_2 = c_2 + 3c_3$
Control Limit CL_s $= c_2\sigma$
Lower Control Limit LCL_s $= (c_2 - 3c_3)\sigma = B_1\sigma$ where $B_1 = c_2 - 3c_3$

Case 2: **σ unknown**

When σ is unknown, it is replaced by its estimate $\bar{s}/c_2$. The control limits are

Upper Control Limit $UCL_R = (c_2+3c_3)\sigma = (c_2 + 3c_3)\dfrac{\bar{s}}{c_2} = B_4\bar{s}$

Central line $CL_R = c2\sigma = c_2\dfrac{\bar{s}}{c_2} = \bar{s}$

Lower Control Limit $LCL_R = (c_2-3c_3)\sigma = (c_2 - 3c_3)\dfrac{\bar{s}}{c_2} = B_3\bar{s}$

Where, $B_4 = (c_2 + 3c_3)/c_2$ and $B_3 = (c_2 - 3c_3)/c_2$

4. Steps in constructing $\overline{X}$ Chart, R-Chart (or S-Chart):

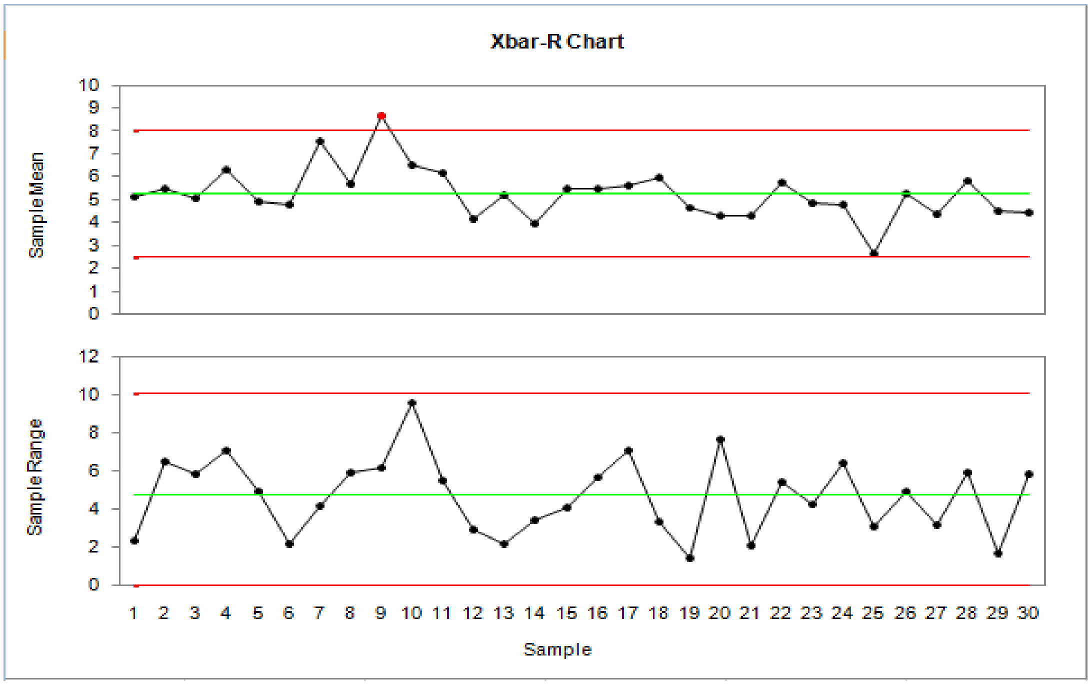
The following steps explain the construction of $\overline{X}$ Chart and R-Chart. Construction of S-Chart is similar to that of R-Chart.

Step 1. *Gather the data.*

a) Select the subgroup size (n). Typical subgroup sizes are 4 to 5. The concept of rational subgrouping should be considered. The objective is to minimize the amount of variation within a subgroup. This helps us "see" the variation in the averages chart more easily.
b) Select the frequency with which the data will be collected. Data should be collected in the order in which it is generated.
c) Select the number of subgroups (k) to be collected before control limits are calculated.
d) For each subgroup, record the individual, independent sample results as shown in table-1.
e) For each subgroup, calculate the subgroup average $\overline{X_i}$, Range (Xmax – Xmin) as shown in table-1.

Step 2. *Plot the data.*

a) Select the scales for the x and y axes for both the $\overline{X}$ Chart and R-Chart.
b) Calculate the Control Limits for $\overline{X}$ Chart and R-Chart and draw the Control lines in the graph.
c) The UCL and LCL lines are drawn as dotted lines and Central line a thick line.
d) Plot the subgroup ranges on the R-Chart and connect consecutive points with a straight line.
e) Plot the subgroup averages on the $\overline{X}$ Chart and connect consecutive points with a straight line.



Step 3: *Interpret the Charts*

Using the Control Chart the process changes and presence of assignable causes are observed and interpreted as detailed in Table-2.

5. Interpretation

Interpretation of $\overline{X}$ Chart:

Table-2

Sl. No.	Situation	Interpretation
1	One or more points beyond the UCL and LCL	Out of control situation
2	Seven (or more) consecutive points above or below the center line	Indication of shift in process average
3	Seven consecutive increasing or decreasing points	Out of Control – presence of trend or tool wearing problem
4	Two out of three beyond two sigma limit on the same side of the center line	Gives a warning of out of control situation – possible shift in process average
5	Four out of five beyond one sigma limit on the same side of the center line	Gives a warning of out of control situation – possible shift in process average
6	Ten out of 11 consecutive points on the same side of the center line	Shift in process average – out of control
7	Twelve out of 14 consecutive points on the same side of the center line	Shift in process average – out of control
8	A series of points "hugging" the centerline	Out of Control Situation – improper estimation of process average
9	A series of points "hugging" the control limits	Out of Control Situation – Shift in the process average above the expected average or below

		the expected average
10	Fourteen consecutive points alternating up and down (saw tooth pattern)	Oscillating process average – proper setting of the average value is necessary.
11	Any nonrandom pattern (such as a cycle)	Cyclical and any other non-random causes of variations have to be identified. Normally they are due to improper estimation of the process average or machine fluctuations.

Interpretation of R Chart

Sl.No.	Situation	Interpretation
1	One or more points beyond the UCL	Out of control situation
2	Seven (or more) consecutive points above Central Line	Indicative of shift in process dispersion to a higher level
3	Seven consecutive increasing points	Out of Control – presence of trend or tool wearing problem
4	Two out of three beyond two sigma limit above the center line	Gives a warning of out of control situation – possible shift in process dispersion
5	Four out of five beyond one sigma limit above the center line	Gives a warning of out of control situation – possible shift in process dispersion
6	Ten out of 11 consecutive points above the center line	Shift in process dispersion – out of control
7	Ten out of 11 consecutive points above the center line	Shift in process dispersion – out of control
8	A series of points "hugging" the centerline	Out of Control Situation – improper estimation of process dispersion
9	A series of points "hugging" the upper control limits	Out of Control Situation – Shift in the process dispersion above the expected dispersion
10	Fourteen consecutive points alternating up and down (saw tooth pattern)	Oscillating process dispersion – proper setting of the average value is necessary.
11	Any nonrandom pattern (such as a cycle)	Cyclical and any other non-random causes of variations have to be identified. Normally they are due to improper estimation of the process average or machine fluctuations.
12	Seven (or more) consecutive points below the center line; Two out of three beyond two sigma limit below the center line; Four out of five beyond one sigma limit below the center line; Ten out of 11 consecutive points below the center line;	Unlike in $\overline{X}$ -Chart, these situation may mean improvement in the process variation. However, the same has to be made sure that the estimate of average process disperion is valid and does not change over time.

Interpretation of S chart is similar to R chart

Trial control limits and control limits:

The trial limits are control limits for analyzing the past data or at the initial stage of establishing control chart, calculated following the procedure explained above. They may require modification before extending them for application for future production and maintenance of stable process. If the data points are within the control limits exhibiting a stable process, we assume the process is in control. These limits are used for future use. Otherwise, we revise the limits by looking at the out-of-control points. Samples for which the plotted values go outside control limits are eliminated for calculation of revised limits. This procedure is continued until all the points are within the control limits or the procedure is free from any assignable cause in obtaining the final control limits. Set up chart for process dispersion (σ-Chart or R-Chart) before working on the $\overline{X}$ -Chart as $\overline{X}$ -Chart requires an estimate of variability from a stable process. If variability is out of control, then a $\overline{X}$ -Chart is not useful. When a $\overline{X}$ -Chart requires a revision, only $\overline{X}$ Chart is revised and the chart for dispersion (σ-Chart or R-Chart) is not revised. This is due to the fact that $\overline{X}$ and R (or 's') are independantly distributed.

Why two control charts – $\overline{X}$ ***-Chart & S-Chart or*** $\overline{X}$ ***-Chart & R-Chart together:***
$\overline{X}$ -Chart chart monitors the ***between*** sample variability and S-Chart or R-Chart monitors the ***within*** sample variability. Hence two charts are necessary to study the process control. Range Chart (R-Chart) indicates a change in the process variability and that of the average chart ($\overline{X}$ -Chart) indicates a change or shift in the level of the process. Hence two charts are constructed simultaneously to see whether the process is working in control or going out of control

Choice between σ-Chart and R-Chart:
If the subgroup sample size is small usually between 2 to 6, R-Chart is appropriate and when sample size is large σ-Chart is appropriate.

6. Table of constants

Control Chart Constants, Sample Sizes to 15

Number of observations in sample, n	Chart for averages			Chart for standard deviations					Chart for ranges				
	Factors for control limits			Factor for central line	Factors for control limits				Factor for central line	Factors for control limits			
	A	A_1	A_2	c_2	B_1	B_2	B_3	B_4	d_2	D_1	D_2	D_3	D_4
2	2.121	3.760	1.880	0.5642	0	1.843	0	3.267	1.128	0	3.686	0	3.267
3	1.732	2.394	1.023	0.7236	0	1.838	0	2.568	1.693	0	4.358	0	2.575
4	1.500	1.880	0.729	0.7979	0	1.808	0	2.266	2.059	0	4.698	0	2.282
5	1.342	1.596	0.577	0.8407	0	1.756	0	2.089	2.326	0	4.918	0	2.115
6	1.225	1.410	0.483	0.8686	0.026	1.711	0.030	1.970	2.534	0	5.078	0	2.004
7	1.134	1.277	0.419	0.8882	0.105	1.672	0.118	1.882	2.704	0.205	5.203	0.076	1.924
8	1.061	1.175	0.373	0.9027	0.167	1.638	0.185	1.815	2.847	0.387	5.307	0.136	1.864
9	1.000	1.094	0.337	0.9139	0.219	1.609	0.239	1.761	2.970	0.546	5.394	0.184	1.816
10	0.949	1.028	0.308	0.9227	0.262	1.584	0.284	1.716	3.078	0.687	5.469	0.223	1.777
11	0.905	0.973	0.285	0.9300	0.299	1.561	0.321	1.679	3.173	0.812	5.534	0.256	1.744
12	0.866	0.925	0.266	0.9359	0.331	1.541	0.354	1.646	3.258	0.924	5.592	0.284	1.716
13	0.832	0.884	0.249	0.9410	0.359	1.523	0.382	1.618	3.336	1.026	5.646	0.308	1.692
14	0.802	0.848	0.235	0.9453	0.384	1.507	0.406	1.594	3.407	1.121	5.693	0.329	1.671
15	0.775	0.816	0.223	0.9490	0.406	1.492	0.428	1.572	3.472	1.207	5.737	0.348	1.652

Statistic	Standards given		Analysis of past data	
	Central line	Limits	Central line	Limits
Average, $\bar{X}$	$\bar{X}'$	$\bar{X}' \pm A\sigma_X'$	$\bar{X}$	$\bar{X} \pm A_1\bar{\sigma}_X$ or $\bar{X} \pm A_2\bar{R}$
Standard deviation, σ_X	$c_2\sigma_X'$	$B_1\sigma_X'$, $B_2\sigma_X'$	$\bar{\sigma}_X$	$B_3\bar{\sigma}_X'$, $B_4\bar{\sigma}_X'$
Range, R	$d_2\sigma_X'$	$D_1\sigma_X'$, $D_2\sigma_X'$	$\bar{R}$	$D_3\bar{R}$, $D_4\bar{R}$

7. **Conclusion**: In this lecture we discussed how to construct control charts for variables in different situations and how to interpret them. We discussed setting up trial and action limits for $\bar{X}$ -Chart, σ-Chart and R-Chart; the different ways of identifying out of control situations using control charts; process in statistical control; when to leave the process alone and when to take corrective measures.